

SYMMETRY GROUPS OF NON-SIMPLY CONNECTED FOUR-MANIFOLDS

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ABSTRACT. Let M be a closed, connected, orientable topological four-manifold with $H_1(M)$ nontrivial and free abelian, $b_2(M) \neq 0, 2$, and $\chi(M) \neq 0$. Then the only finite groups which admit homologically trivial, locally linear, effective actions on M are cyclic.

The proof uses equivariant cohomology, localization, and a careful study of the first cohomology groups of the (potential) singular set.

1. INTRODUCTION

This paper can be viewed as a sequel to [11], where, following a conjecture of Edmonds [8], we showed that if M is a simply-connected four-manifold with $b_2(M) \geq 3$, and G is a finite or compact Lie group which acts effectively, locally linearly, and homologically trivially on M , then G must be isomorphic to a subgroup of $S^1 \times S^1$.

Here we consider the more general situation in which $H_1(M; \mathbb{Z})$ is free abelian of arbitrary rank. We prove:

Theorem 1.1 (Main Theorem). *Let M be a closed, connected, orientable topological four-manifold with $H_1(M)$ nontrivial and free abelian, $b_2(M) \neq 0, 2$, and $\chi(M) \neq 0$. Then the only finite groups which admit homologically trivial, locally linear, effective actions on M are cyclic.*

The assumptions that $\chi(M) \neq 0$ and $b_2(M) \neq 2$ are necessary, as familiar examples of actions on T^4 and the product of S^2 with any closed, oriented surface make clear. Although we do not explicitly indicate so each time, many arguments extend to the case where torsion in $H_1(M)$ is relatively prime to the orders of the groups involved.

The analysis of [11] was based on a comparison of the Borel equivariant cohomology of M with that of its singular set Σ , and an important ingredient in understanding Σ was Edmonds's observation [7] that in the simply-connected case, the fixed-point set of any cyclic group action consists only of isolated points and spheres, with no surfaces of higher genus. Using local considerations, it becomes possible to assemble these fixed sets to gain a rather explicit

description of Σ for a (potential) action of a larger finite group, and eventually rule out the nonabelian ones. The arguments were homological in nature, and hence extend to the case where $H_1(M; \mathbb{Z}) = 0$.

Our current situation differs in several important ways. First, algebraic considerations now allow (indeed, force) fixed-point sets to contain surfaces of higher genus, and a purely combinatorial assembly of Σ from its components is no longer feasible. Moreover, when all cohomology of M was concentrated in even degrees, nearly complete information about the cohomology of Σ could ultimately be extracted from the map $H^2(M) \rightarrow H^2(\Sigma)$. In contrast, for a cyclic group C_p , the restriction $H^1(M) \rightarrow H^1(\text{Fix}(C_p))$ has half rank at best, so it now requires more work to understand and exploit the induced action of a group G on the cohomology of its singular set. Finally, in [11], gaps in odd degrees frequently led the spectral sequences involved in computing $H_G^*(M)$ to collapse for formal reasons. They require more careful attention here, and in some cases, our understanding of the differentials remains incomplete. On the positive side, $H^1(\Sigma)$, to the extent that it can be detected, ultimately imposes considerable rigidity on a group action.

The structure of the paper is as follows: In section 2, we collect and provide references for our main technical tools: the Lefschetz Fixed-point Theorem, Borel equivariant cohomology, and the Localization Theorem of Borel, tom Dieck, Hsiang, and Quillen. In section 3, we consider the collapsing of the Borel Spectral Sequence for various groups and derive some immediate consequences, including generalizations of a few useful results of [7]. In section 4, we discuss, modify, and generalize some (slightly flawed) arguments of [2] to analyze the action of a rank two abelian group on the fixed set of a cyclic subgroup, and ultimately rule out actions by groups of rank two. Finally, in section 5, we consider actions of metacyclic and quaternion groups, rule them out (using an application of localization which might be of independent interest), and gather the pieces to prove the main theorem.

Our standing assumptions throughout the paper are that M is a closed, connected, orientable topological four-manifold, and G is a finite group acting effectively, homologically trivially, and locally linearly (“HTLL”) on M . We generally use C_n to denote $\mathbb{Z}/n\mathbb{Z}$ when regarding it as a (transformation) group, and \mathbb{Z}_n when regarding it as a ring or field of coefficients. Cohomology should be assumed to be ordinary singular cohomology using integer coefficients in the absence of indications to the contrary.

2. GENERAL CONSIDERATIONS AND BACKGROUND

Let $\langle g \rangle = C_n$. Local linearity implies that $\text{Fix}(g)$ is a locally flat submanifold of M , and if the action preserves orientation on M (as we shall always assume), the fixed-point set must have even codimension, and hence consist of a union of points and surfaces. When n is odd, each surface component must be orientable (see [4]). For locally linear actions, we also have

a strong form of the Lefschetz fixed-point theorem: $\chi(\text{Fix}(g)) = \sum (-1)^i \text{Trace}(g^*|_{H^i(M)})$. If g acts trivially on homology, it follows that $\chi(\text{Fix}(g)) = \chi(M)$.

The Borel construction $M_G = M \times_G EG$ defines a fibration $M \rightarrow M_G \rightarrow BG$, and the Leray-Serre spectral sequence of this fibration $E_2^{ij}(M) = H^i(G; H^j(M)) \Rightarrow H^*(M_G)$ is one of our fundamental tools. Henceforth we refer to it as the *Borel Spectral Sequence* (BSS). It is common practice to refer to the groups $H_G^*(M) := H^*(M_G)$ as *(Borel) equivariant cohomology groups*. For more background, see any of [3], [14], [4], or [9].

Several facts about these groups and the spectral sequence are of particular importance for us:

- (1) The BSS is equipped with a well-behaved $H^*(G)$ -algebra structure. In particular, if $H^*(M)$ is torsion-free and G acts trivially on it, then $E_2 \cong H^*(M) \otimes H^*(G)$.
- (2) Let $\Sigma := \{x \in M \mid G_x \neq \{e\}\}$ denote the *singular set* of the action. Then restriction induces an isomorphism $H_G^*(M) \rightarrow H_G^*(\Sigma)$ in dimensions $* > 4$ (or more generally, greater than the dimension of the manifold under consideration).
- (3) The functor H_G is natural with respect to maps of groups and G -spaces.

Finally, recall the Localization Theorem (cf. [14, 1]): Let a finite group G act (reasonably) on a space X , let S be a multiplicatively closed, central subset of $H^*(G)$, and let $\Sigma_S = \{x \in X \mid S \cap \ker(i^* : H^*(G) \rightarrow H^*(G_x)) = \emptyset\}$. Then inclusion induces an $S^{-1}H^*(G)$ -module isomorphism $S^{-1}H_G^*(X) \rightarrow S^{-1}H_G^*(\Sigma_S)$. Note that if S includes an element of degree d , then $S^{-1}H_G^*(X)$ and $S^{-1}H_G^{*+d}(X)$ become formally identified, so after localization, degrees must be interpreted (at most) modulo d .

3. COLLAPSING OF THE SPECTRAL SEQUENCE

Poincaré duality and the presence of a nonempty fixed-point set together impose strong restrictions on the differentials in the spectral sequence:

Proposition 3.1. *Let $R = \mathbb{Z}$ or \mathbb{Z}_p . Suppose a finite group G acts on a closed four-manifold M , with $H_*(M; R)$ R -torsion-free, trivially on $H^*(M; R)$, and with at least one fixed point. Then the differential d_2 in the BSS vanishes in all of row 4, all of row 3, all of row 1, and on those classes in row 2 which are products of one-dimensional classes. Indeed, the only potentially nonzero differential d_r (for $r \geq 2$) is $d_2^{i,2}$, and its values are determined by $d_2^{0,2} : H^0(G; H^2(M)) \rightarrow H^2(G; H^1(M))$. If $d_2 = 0$, then the spectral sequence collapses.*

In particular, if G is cyclic of prime order, then this spectral sequence collapses.

Proof. Let $x \in M^G$. The restriction homomorphism $j^* : H^*(M) \rightarrow H^*(M \setminus \{x\})$ is zero in dimension four, and an isomorphism in other dimensions. It follows that the corresponding

map of spectral sequences $E_2^{i,j}(M) \rightarrow E_2^{i,j}(M \setminus \{x\})$ is trivial when $j = 4$, and an isomorphism otherwise. Factoring through these maps, using naturality of the differentials, shows that $d_2^{i,4} = 0$ for all i .

Similarly, factoring through the map of spectral sequences $E_2^{i,j}(M, \{x\}) \rightarrow E_2^{i,j}(M)$ shows that $d_2^{i,1} = 0$ for all i .

If $H_*(M)$ is R -torsion-free, then the universal coefficient theorem and Poincaré duality together yield a nonsingular intersection pairing $H^*(M) \otimes H^{4-*}(M) \rightarrow H^4(M)$. So for any generator $u \in E_2^{0,3} = H^0(G; H^3(M))$, there is $v \in E_2^{0,1} = H^0(G; H^1(M))$ such that uv generates $E_2^{0,4} = H^0(G; H^4(M))$. But $d_2(uv) = 0 = ud_2(v) - d_2(u)v$. Since $d_2(v) = 0$, $d_2(u)v = 0$. But the E_2 term of the spectral sequence is a free $H^*(G)$ -algebra, so $d_2(u) = 0$, as well.

Finally, if $G = C_p$, Adam Sikora [12, 3.2(i), 3.11, 3.13] has shown¹ that the terms E_r^{ij} of the spectral sequence satisfy a form of Poincaré duality for each $r \geq 2$ which implies that $\text{rk } E_3^{2,1} = \text{rk } E_3^{2,3}$, and hence that $d_2^{0,2} = 0$.

Vanishing of d_3 on row 3 follows from the fact that generators of $H^3(M)$ are dual to those of $H^1(M)$, and all remaining differentials are easily accounted for by factoring through $E(M - \{x\})$ and $E(M, \{x\})$, as appropriate. \square

Corollary 3.2. *Let p be prime. In the above situation (in particular, with $\text{Fix}(G) \neq \emptyset$), if $G = C_p \times C_p$, the BSS collapses with integral coefficients.*

Proof. Recall (cf. [10]) that

$$H^*(C_2 \times C_2; \mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha_2, \beta_2] \otimes P[\mu_3]}{\langle 2\alpha = 2\beta = 2\mu = 0, \mu^2 = \alpha\beta^2 + \alpha^2\beta \rangle},$$

while for p odd,

$$H^*(C_p \times C_p; \mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha_2, \beta_2] \otimes \bigwedge[\mu_3]}{\langle p\alpha = p\beta = p\mu = 0 \rangle}.$$

In both cases, the elements α_2 and β_2 arise as Bocksteins of elements of $H^1(C_p; \mathbb{Z}_p) = \text{Hom}(C_p, \mathbb{Z}_p)$, so it is easy to check that every $x \in H^2(C_p \times C_p; \mathbb{Z})$ is detected by restriction to some cyclic subgroup $C_p < G$.

¹Sikora's result is stated for the Leray spectral sequence of the map $X_G \rightarrow B_G$, which is defined in terms of sheaf cohomology. But according to Bredon [5, III.1.1], whenever X is a CW complex, Φ is a paracompactifying family of closed sets, and \mathcal{A} is a sheaf of local coefficients, there is a natural isomorphism $H_\Phi^*(X; \mathcal{A}) \rightarrow {}_\Delta H_\Phi^*(X; \mathcal{A})$ between sheaf and ordinary singular cohomology groups. This isomorphism covers all cases of interest to us here. In particular, since the Leray and the Serre spectral sequences have isomorphic E_2 -terms and abutments, and the Leray spectral sequence collapses, it follows for dimension reasons that the Serre spectral sequence must collapse, allowing us to bypass any technical verification that the two spectral sequences are themselves isomorphic.

Hence, if $d_2^{0,1}(xb) = ya \neq 0$ for some $x \in H^0(C_p \times C_p)$, $b \in H^2(M)$, $y \in H^2(C_p \otimes C_p)$, and $a \in H^1(M)$, then there is a restriction r^* to some cyclic subgroup $C_p < C_p \times C_p$ so that $r^*(y) \neq 0$. But d_2 commutes with r^* , so for the cyclic group action, $d_2(xb) \neq 0$, contradicting Proposition 3.1. \square

It follows from the Universal Coefficient Theorem that all torsion in the cohomology ring is determined by $H_1(M)$. Henceforth we assume that $H_1(M)$ is torsion-free, and note some consequences of Lemma 3.1

Corollary 3.3. *Suppose M is a four-manifold with $H_1(M)$ torsion-free, and suppose C_p , where p is prime, acts HTLL on M . If $\chi(M) \neq 0$, then $b_1(\text{Fix}(C_p)) = 2b_1(M)$, and $b_2(\text{Fix}(C_p)) = 2 + b_2(M)$.*

Proof. In this situation, it follows from the Lefschetz fixed-point theorem that $\chi(\text{Fix}(C_p)) = \chi(M) \neq 0$. In particular, $\text{Fix}(C_p) \neq \emptyset$.

Since $H_G^*(M) \cong H_G^*(F)$ in high degrees, it follows easily that $b_1(F) = 2b_1(M)$, $b_0(F) + b_2(F) = 2 + b_2(M)$. \square

Recall that a *pseudofree* action of a finite group is one in which the singular set consists only of isolated points. We note in passing:

Corollary 3.4. *Suppose M is a four-manifold with $H_1(M)$ torsion-free, and suppose $b_1(M) > 0$. Then M admits no pseudofree, homologically trivial group actions if $\chi(M) \neq 0$. If $\chi(M) = 0$, then the only possible pseudofree actions are actually free actions.*

The following proposition is due to Edmonds[7] in the simply-connected case. Generalization to the case at hand presents no new difficulties:

Proposition 3.5. *Let M be a four-manifold with $H_1(M; \mathbb{Z})$ torsion-free, with an HTLL action by $G = C_p$ (p prime). If F' is any proper subset of the fixed-point set F , then the restriction map $H^2(M; \mathbb{Z}_p) \rightarrow H^2(F'; \mathbb{Z}_p)$ is surjective.*

Proof. Without loss of generality, we assume that F contains at least two points (and hence that the BSS collapses). Let $y \in F - F'$, and $x \in F$. As in [7], the map of spectral sequences $E(M - y, x) \rightarrow E(F - y, x)$ converges to an isomorphism in degrees > 4 , and in particular, in degree 6. The hypotheses on $H^1(M)$ imply that $H^5(G; H^1(M; \mathbb{Z}))$ and $H^3(G; H^3(M; \mathbb{Z}))$ vanish, so $H^6(M_G - y_G; x_G) \cong H^4(G; H^2(M - y, x))$. Meanwhile, the Leray-Hirsch theorem implies that the restriction $H^6(F_G - y_G, x_G) \rightarrow E_\infty^{4,2}(F - y, x)$ is onto. Edmonds's commutative diagram [7, page 115] implies the result. \square

Remark. This result actually generalizes to the case when C_p acts nontrivially, but without cyclotomic-type summands, on $H^*(M)$. One appeals to a stronger version of Sikora's result to

show the spectral sequence still collapses, then shows that $H^5(G; H^1(M))$ and $H^3(G; H^3(M))$ make no contribution to $H_G^6(M)$. However, simple constructions show the result to be false when cyclotomic actions on $H^1(M)$ are permitted.

Corollary 3.6 ([7]). *In the above situation:*

- (1) *If F is not purely 2-dimensional, then the surface components of F represent independent elements of $H_2(M; \mathbb{Z}_p)$.*
- (2) *If F is purely two-dimensional, with k components, then the surfaces in F span a subspace of $H_2(M; \mathbb{Z}_p)$ of dimension at least $k - 1$, with any $k - 1$ components representing independent elements.*
- (3) *If $\text{Fix}(G)$ has more than two components, then the surface components all represent different elements of $H_2(M)$, and hence can not be interchanged by a homologically trivial symmetry.*

Proposition 3.7. *Let M be a closed, oriented four-manifold with nonzero Euler characteristic, $b_2(M) \neq 0$, $H_1(M; \mathbb{Z})$ torsion-free, and suppose g is a homologically trivial involution on M . Then each component of $\text{Fix}(g)$ is orientable.*

Proof. When $b_1(M) = 0$, this was remarked in [7, p.114]. When $b_1(M) > 0$, the hypotheses imply that (using mod 2 Betti numbers) $b_0(\text{Fix}(g)) + b_2(\text{Fix}(g)) \geq 3$, so $\text{Fix}(g)$ must have at least two components, and by Proposition 3.5, it follows that for any two-dimensional component $F' \subset \text{Fix}(g)$, the \mathbb{Z}_2 -fundamental class $[F']$ is nontrivial in $H_2(M; \mathbb{Z}_2)$. But $H_1(M; \mathbb{Z})$ is torsion-free, so $H_2(M; \mathbb{Z}_2) \cong H_2(M; \mathbb{Z}) \otimes \mathbb{Z}_2$. It follows that $[F]$ is nontrivial in $H_2(M; \mathbb{Z})$, so F is orientable.

□

4. THE ACTION OF G ON Σ AND ITS CONSEQUENCES FOR RANK TWO GROUPS.

Whenever $C_p \triangleleft G$, there is an induced action of G/C_p on $\text{Fix}(C_p)$. In this section we consider the extent to which this action can be nontrivial.

Consider the following basic example. Let $G = C_2 \times C_2 = \langle s, t \rangle$ act linearly on S^2 . Then $\text{Fix}(s)$ consists of two points, and the t -action interchanges them. (Examples of actions on four-manifolds can be constructed in a similar vein, e.g the action on S^2 of any subgroup of $SO(3)$ defines obvious actions on S^2 bundles over surfaces, and $C_3 \times C_3$ acts linearly on \mathbb{CP}^2 , with permutation actions on the fixed sets of cyclic subgroups.)

A slightly flawed argument in [2, Lemma 5.2] leads to the erroneous assertion that in such situations, G/C_p must act trivially on $H^*(M^{C_p})$. The idea boils down to the following: In large dimensions, $H_{C_p}^*(M) \cong H_{C_p}^*(M^{C_p})$. The action on the terms of the spectral sequence for the C_p action on M is trivial; so the action on $H^*(M^{C_p})$ must be trivial as well. The

oversight in the argument is that $E(M)$ converges to the associated graded of $H_{C_p}(M)$, not to the equivariant cohomology itself. Even with field coefficients, the action can be trivial on the associated graded, but nontrivial on the module. But the idea of viewing G as acting on $E(M)$ is still useful.

In the case of S^2 , let S denote the subgroup $\langle s \rangle$. The fact that the spectral sequence has only two rows yields the short exact sequence:

$$0 \rightarrow E_{\infty}^{4,0} \rightarrow H_S^4(S^2) \rightarrow E_{\infty}^{2,2} \rightarrow 0.$$

The collapsing of the spectral sequence for the S -action, together with the isomorphism induced by the inclusion of the fixed-point set, means that the sequence becomes:

$$0 \rightarrow H^4(C_2; H^0(S^2)) \rightarrow H_S^4(\text{Fix}(S)) \rightarrow H^2(C_2; H^2(S^2)) \rightarrow 0,$$

and finally, since the BSS for the fixed set has only one row, $H_S^4(\text{Fix}(S)) \cong H^4(C_2) \otimes H^0(\text{Fix}(S))$, so we get

$$0 \rightarrow \mathbb{Z}_2 \otimes H^0(S^2) \rightarrow \mathbb{Z}_2 \otimes H^0(\text{Fix}(S)) \rightarrow \mathbb{Z}_2 \otimes H^2(S^2) \rightarrow 0.$$

Now, S is (obviously) central in G , so the action of G/S on $\text{Fix}(S)$ is S -equivariant. (Equivalently, G/S acts on the fibration $X_{C_p} \rightarrow BC_p$ by fiber-preserving maps of fibrations.) It follows that G/S acts on the spectral sequence, so we have a short exact sequence of $\mathbb{Z}[G/S]$ -modules; since everything is 2-torsion, it is a SES of $\mathbb{Z}_2[G/S]$ -modules, with trivial modules on the ends.

Maschke's Theorem (see [6]) provides a sufficient condition for triviality on the ends of such a sequence to imply triviality in the middle: the $\mathbb{Z}_2[G/S]$ -module $H^4(C_2; H^0(\text{Fix}(S)))$ is completely reducible whenever $|G/S|$ is relatively prime to $\text{char } \mathbb{F}_2 = 2$. (Of course, in this case, it is not.) In this case, the image of the map $\mathbb{Z}_2 \otimes H^0(S^2) \rightarrow \mathbb{Z}_2 \otimes H^0(\text{Fix}(S))$ is the subgroup generated by $[x_0] + [x_1]$. The actions on both the subgroup and the resulting quotient are trivial.

This argument also breaks down for other reasons when C_p is not central in G , as is easily seen by inspection of standard dihedral group actions on S^2 . We will use localization to study these actions more closely in Section 5.

With these considerations in mind, let $G = C_p \times C_p = \langle g \rangle \times \langle h \rangle$. Recall that M is a four-manifold with $H_1(M; \mathbb{Z})$ torsion-free, and the action of G is HTLL. Assume $\chi(M) \neq 0$.

By Lemma 3.1, the BSS for the action of $\langle g \rangle$ collapses, so $H_{\langle g \rangle}(M; \mathbb{Z}) \cong H^*(\langle g \rangle) \otimes H^*(M)$ as $H^*(\langle g \rangle)$ -modules. The G -isomorphism $H_{\langle g \rangle}^5(M) \cong H_{\langle g \rangle}^5(\text{Fix}(g))$ then gives us a short exact sequence of $G/\langle g \rangle$ -modules

$$0 \rightarrow H^4(C_p; H^1(M)) \rightarrow H_{\langle g \rangle}^5(\text{Fix}(g)) \rightarrow H^2(C_p; H^3(M)) \rightarrow 0,$$

or equivalently,

$$0 \rightarrow \mathbb{Z}_p \otimes H^1(M; \mathbb{Z}) \rightarrow \mathbb{Z}_p \otimes H^1(\text{Fix}(g); \mathbb{Z}) \rightarrow \mathbb{Z}_p \otimes H^3(M; \mathbb{Z}) \rightarrow 0,$$

in which $\langle h \rangle$ acts trivially on all terms except possibly the middle.

Since $\text{Fix}(g)$ is orientable (assuming $b_2(M) \neq 0$ when $p = 2$), $H^1(\text{Fix}(g); \mathbb{Z})$ is free abelian. By the classification of integral C_p -representations [6, 74.3],

$H^1(\text{Fix}(g); \mathbb{Z})$ splits into a sum of $\langle h \rangle$ -modules of trivial, permutation, and cyclotomic types (with no exotic ideal classes; cf. [13]).

On the other hand, the classification of $\mathbb{Z}_p[\langle h \rangle]$ -modules is elementary (cf. [12, 2.1]): Each splits essentially uniquely as a sum of indecomposables, and each indecomposable is a cyclic $\mathbb{Z}_p[t]$ -module of the form $\mathbb{Z}_p[t]/(t-1)^i$, where $1 \leq i \leq p$. If an indecomposable $\mathbb{Z}_p[\langle h \rangle]$ -module M arises as $\mathbb{Z}_p \otimes N$, where N is \mathbb{Z} -free, then M must have $i = 1, p-1$, or p . And if M then has both a submodule and a corresponding quotient module with trivial $\langle h \rangle$ -action, then either $p = 2$, or M itself has trivial $\langle h \rangle$ -action. It follows that:

- (1) When $p = 2$, $H^1(\text{Fix}(g); \mathbb{Z}_p)$ may contain summands on which h acts by permutation, but
- (2) When $p \geq 3$, the action of $\langle h \rangle$ on $H^1(\text{Fix}(g); \mathbb{Z}_p)$ must be trivial.

Lemma 4.1. *Let M be a four-manifold whose first homology group is torsion-free, with $\chi(M) \neq 0$, and suppose M admits a homologically trivial, locally linear $C_p \times C_p$ -action, with p an odd prime. Then the singular set of the action consists of isolated points, chains of two-spheres, and isolated tori only. Moreover, each non-identity group element fixes exactly $b_1(M)$ tori.*

Proof. Choose any pair of generators g, h for the group. The hypotheses guarantee that the BSS for the $\langle g \rangle$ action collapses, so $\text{Fix}(g)$ consists of points and orientable surfaces. Since the $\langle h \rangle$ -action on $H^1(\text{Fix}(g))$ must be trivial, the Lefschetz Fixed-point theorem rules out the existence of components of genus > 1 . If T is a torus component of $\text{Fix}(g)$, and F is a component of $\text{Fix}(h)$ meeting T , then the h -action on T fixes a point, and hence must fix the whole torus. This is impossible, since G cannot act faithfully on a two-dimensional slice disk. So $\langle h \rangle$ acts freely on T . The torus count then follows from the fact that $b_1(\text{Fix}(g)) = 2b_1(M)$. □

Theorem 4.2. *Let $G = C_p \times C_p$, where p is an odd prime, and let M be a closed topological four-manifold such that $H_1(M; \mathbb{Z})$ is torsion-free, with $\chi(M) \neq 0$. If G acts HTLL on M , then $b_1(M) = 0$.*

Proof. Assume first that $\text{Fix}(G) \neq \emptyset$; then it consists of a finite set of points, and by Corollary 3.2, the BSS collapses with integer coefficients. Localizing with respect to the

set $S \subset H^*(G)$ generated by $\alpha_2, \beta_2 \in H^2(G)$ yields an isomorphism $S^{-1}(H^*(G) \otimes H^*(M)) \cong S^{-1}(H^*(G) \otimes H^*(\text{Fix}(G)))$. But $S^{-1}H^*(G)$ is a \mathbb{Z}_2 -graded vector space of dimension 1 in both even and odd dimensions. It follows that $\sum_{i=0}^4 b_i(M) = |\text{Fix}(G)|$. It also follows from a simple analysis of the singular set that $\langle h \rangle$ acts trivially on $H^*(\text{Fix}(G))$, so by the Lefschetz theorem, $\chi(M) = \chi(\text{Fix}(g)) = \chi(\text{Fix}(G)) = |\text{Fix}(G)|$. Hence $\sum_{i=0}^4 b_i(M) = \chi(M)$, so the odd Betti numbers of M must vanish.

Henceforth we assume (for contradiction) that $\text{Fix}(G) = \emptyset$, and that $b_1(M) > 0$. Every nontrivial cyclic subgroup $\langle g \rangle$ fixes at least one torus. If g were also to fix more than one copy of S^2 , then by Corollary 3.6, all surface components of $\text{Fix}(g)$ represent different elements of $H_2(M)$, so each must be G -invariant. If g fixes a single S^2 , then it is obviously G -invariant. But a G -invariant two-sphere contains a fixed point. The only remaining possibility is that for every $g \in G$, $\text{Fix}(g)$ consists of $\chi(M)$ isolated points, freely permuted by $G/\langle g \rangle$, and $b_1(M)$ tori, each equipped with a free $G/\langle g \rangle$ -action. Since $G/\langle g \rangle$ freely permutes the isolated points, $p \mid \chi(M)$. It follows that $b_2(M) \geq p \geq 3$.

Recall that each cohomology class in $H^2(G; \mathbb{Z})$ restricts nontrivially to the cohomology of some cyclic subgroup $C_p \subset G$ – although the generator $\mu \in H^3(G; \mathbb{Z})$ does not. Consider the term $E_2^{0,2}$ in the spectral sequence for the G -action. Since $d_2^{0,2} : H^0(G; H^2(M; \mathbb{Z})) \rightarrow H^2(G; H^1(M; \mathbb{Z})) \cong H^2(G) \otimes H^1(M)$, any nontriviality of this differential would be detected by a cyclic subgroup. Hence by Proposition 3.1, $d_2^{0,2} = 0$. The differential $d_3^{0,2}$ could be nonzero, but its target $H^3(G; H^0(M))$ is one dimensional. Hence $[E_2^{0,2} : E_\infty^{0,2}] \leq p$. Since $b_2(M) \geq 3$, there must exist generators $x, y \in H^2(M)$ which survive to E_∞ and such that xy generates $H^4(M; \mathbb{Z})$. Hence $E_2^{0,4} = E_\infty^{0,4}$.

By the multiplicative structure of the spectral sequence, it follows as in [1, 5.4.1] that the $E_\infty^{*,4}$ row is a free $H^*(G)$ -module, and that the filtration of $H_G^*(M; \mathbb{Z})$ associated with the spectral sequence yields an exact sequence

$$0 \rightarrow \mathcal{F}_3 H_G^*(M; \mathbb{Z}) \rightarrow H_G^*(M; \mathbb{Z}) \rightarrow E_\infty^{*,4} \rightarrow 0.$$

Localizing with respect to $S = \langle \alpha, \beta \rangle$, it follows that $\text{Fix}(G) \neq \emptyset$, a contradiction.

□

Unsurprisingly, the case $p = 2$ requires special treatment. (Recall that $C_2 \times C_2$ acts on S^2 , and hence on the product of S^2 with any surface.)

Proposition 4.3. *Let $G = C_2 \times C_2$ act HTLL on M , where $H^*(M; \mathbb{Z})$ is torsion-free. If $\chi(M) \neq 0$ and $b_2(M) \neq 0, 2$, then the BSS collapses with \mathbb{Z} coefficients.*

Proof. Since $H^2(G; \mathbb{Z})$ is detected by cyclic subgroups, d_2 vanishes. For d_3 , we adapt a calculation from [8, 6.1]: Let $u \in H^2(M)$. If $u^2 \neq 0 \pmod{2}$, then $0 = d_3(u^3) = 3d_3(u) \cdot u^2$, so $d_3(u) = 0$. On the other hand, if $u^2 = 0 \pmod{2}$, then $b_2(M) \neq 1$, and since we assume

$b_2(M) \neq 2$, there exists another generator $v \in H^2(M)$, linearly independent of u , so that $uv = 0$. Then $0 = d_3(uv) = d_3(u) \cdot v + u \cdot d_3(v)$. But E_3 is a free $H^*(G)$ -module, so by linear independence, $d_3(u) = d_3(v) = 0$. Hence d_3 vanishes on $H^2(M)$. Since $H^4(M)$ is generated by products of two-dimensional classes, d_3 vanishes on H^4 , as well. Finally, let u be a generator of $H^3(M)$. There exists $v \in H^1(M)$ such that uv generates $H^4(M)$, so $d_3(uv) = 0$. But $d_3(u) \cdot (v) + u \cdot d_3(v) = 0$, and since $d_3(v) = 0$, $d_3(u) = 0$, as well.

Finally, d_4 and d_5 vanish on $H^1(M)$ and $H^2(M)$ for dimension reasons, and then on the rest of $H^*(M)$ by Poincaré duality. \square

Corollary 4.4. *In the above situation, Localization shows that $\dim H^*(M^G; \mathbb{Z}) = \dim H^*(M; \mathbb{Z})$.*

Theorem 4.5. *Let $G = C_2 \times C_2$, and let M be a closed topological four-manifold such that $H^*(M; \mathbb{Z})$ is torsion-free, with $\chi(M) \neq 0$. and $b_2(M) \neq 0, 2$. If G acts HTLL on M , then $b_1(M) = 0$.*

Proof. Let $G = \langle g \rangle \times \langle h \rangle$. In the proof of Theorem 4.2 for odd p , we knew the G -action on $H^1(\text{Fix}(g))$ was trivial, and in the presence of a fixed point, the result followed immediately from the Lefschetz theorem. Here, if $G = \langle g \rangle \times \langle h \rangle$, we have not ruled out the possibility that h might act by permutations on part of $H^1(\text{Fix}(g))$.

It is now a standard application of localization (cf. [14, III.4.16]) that for any C_p -space X such that $\dim_{\mathbb{Z}_p} \bigoplus H^k(X; \mathbb{Z}_p)$ is finite, we have

$$\dim_{\mathbb{Z}_p} \bigoplus H^k(X^{C_p}; \mathbb{Z}_p) \leq \dim_{\mathbb{Z}_p} \bigoplus H^k(X; \mathbb{Z}_p),$$

with equality if and only if C_p acts trivially on $H^*(X)$ and the BSS collapses.

Separate $\text{Fix}(g)$ into its $\langle h \rangle$ -orbits of path-components, say X_1, \dots, X_k . Each is either an oriented surface, a pair of such surfaces interchanged by h , a point, or a pair of points. Then

$$\begin{aligned} |\text{Fix}(G)| &= \sum_{i=1}^k |\text{Fix}(h, X_i)| \\ &\leq \sum_{i=1}^k (\dim_{\mathbb{Z}_p} \bigoplus H^j(X_i; \mathbb{Z}_p)) \\ &= \dim_{\mathbb{Z}_p} \bigoplus H^i(\text{Fix}(g); \mathbb{Z}_p) \\ &= \sum b_j(M) \\ &= |\text{Fix}(G)|. \end{aligned}$$

It follows that for each individual X_i , $|\text{Fix}(h, X_i)| = \sum(b_j(X_i))$. By the Lefschetz Theorem, this is only possible if $\text{Trace}(h|_{H^1(X_i)}) = -b_1(X_i)$. But the trace of a permutation representation is zero. Hence $b_i(X_i) = 0$ for all $X_i \subset \text{Fix}(g)$. So $b_1(\text{Fix}(g)) = 0$, and finally, $b_1(M) = 0$.

□

5. NONABELIAN GROUPS

Recall that the *rank* of a finite group G is the largest n such that there exists a prime p with $(C_p)^n \subset G$. It follows from the results of the last section that if M is a closed, oriented four-manifold with $H_1(M)$ nontrivial and torsion-free, $\chi(M) \neq 0$, $b_2(M) \neq 0, 2$, then any finite group which acts HTLL on M must have rank one. In [11], we analyzed minimal nonabelian groups and saw in particular that if G is a nonabelian rank one finite group such that every proper subgroup of G is abelian, then G is either a metacyclic group of the form $C_p \rtimes C_{q^n}$, where p and q are prime, and C_{q^n} acts on C_p via an order q group automorphism, or Q_8 , the order 8 quaternion group.

Our remaining arguments rely on localization to yield information about the action of a group on the fixed set of a normal subgroup. As applications in the literature seem to focus mainly on abelian groups, we begin with a “baby” example for motivation. Let p be an odd prime, and consider a standard linear action of the dihedral group $D_p = \langle a, b \mid a^p = b^2 = 1, bab^{-1} = a^{-1} \rangle$ on S^2 . Each cyclic subgroup fixes two points; each involution interchanges the points in $\text{Fix}(a)$, and the action of $\langle a \rangle = C_p$ permutes the fixed sets of the various 2-subgroups: $a \cdot \text{Fix } t = \text{Fix}(ata^{-1})$.

The integral cohomology of D_p is periodic of period 4, vanishing in odd dimensions, cyclic of order 2 in dimensions equal to 2 mod 4, and cyclic of order $2p$ in dimensions divisible by 4. (This is well-known, but also follows from our general calculation for metacyclic groups below). With this in mind it is easy to see that the BSS for the D_p action on S^2 collapses, so $H_{D_p}^*(S^2; \mathbb{Z})$ is a free $H^*(D_p)$ module on generators of degree zero and two.

In the case of our D_p action, the periodicity generator u is detected by restriction to both C_p and C_2 , so if S is the multiplicative set generated by u , then Σ_S is the entire singular set. Hence localization tells us nothing we did not already know by simply considering the (non-localized) equivariant cohomology groups in degrees $d > 2$. However, if S is generated instead by $2u$, then $\Sigma_S = \text{Fix}(C_p)$, and $S^{-1}H^*(D_p)$ is a \mathbb{Z}_4 -graded module with \mathbb{Z}_p in degree four, and zeroes otherwise. It follows that as a $S^{-1}H^*(D_p)$ -module, $S^{-1}H_{D_p}^*(\text{Fix}(C_p))$ has one \mathbb{Z}_p in each even degree.

We know (geometrically) that $\text{Fix}(C_p)$ is a 0-dimensional manifold, equipped with an action of $D_p/C_p \cong C_2$. It follows that $H^0(\text{Fix}(C_p))$ is a \mathbb{Z} -free $\mathbb{Z}[C_2]$ -module, and hence

a sum of modules of trivial, permutation, and cyclotomic types. But one checks that $H^*(D_p; \mathbb{Z}[D_p/C_p])$ has generators in every even degree, and $H^*(D_p; \mathbb{Z}[-1])$ has generators in degrees $d \cong 2 \pmod{4}$. Localization therefore detects the fact that the action of D_p/C_p on $\text{Fix}(C_p)$ is nontrivial: either entirely by permutations, or with equally many trivial and cyclotomic components. It is simple to rule out the latter case with other considerations, but even in the absence of such considerations, note the consequence that the trace of the C_2 -action on $H^0(\text{Fix}(C_p))$ is zero.

Now consider the case of a metacyclic group $G = \langle a, b \mid a^p = 1 = b^{q^n}, bab^{-1} = b^r \rangle$, where p and q are prime, and $r^q \equiv 1 \pmod{p}$. The Lyndon-Hochschild-Serre spectral sequence of the group extension $1 \rightarrow C_p \rightarrow G \rightarrow C_{q^n} \rightarrow 1$ has $H^i(C_{q^n}; H^j(C_p; \mathbb{Z})) \Rightarrow H^{i+j}(G; \mathbb{Z})$. Since p and q are relatively prime, $H^i(C_{q^n}; H^j(C_p; \mathbb{Z})) = 0$ whenever i and j are both positive. And in general, $H^0(C_{q^n}, X) \cong X^{C_{q^n}}$ for any C_q -module X . The action of C_{q^n} on $H^*(C_p; \mathbb{Z})$ is determined by its action on the generator t of $H^2(C_p)$, and $b \cdot t = rt$. Hence

$$H^0(C_{q^n}; H^j(C_p; \mathbb{Z})) = \begin{cases} \mathbb{Z} & \text{if } j = 0, \\ \mathbb{Z}_p & \text{if } j > 0 \text{ and } 2q \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

Of course, $H^i(C_{q^n}; H^0(C_p; \mathbb{Z})) = \mathbb{Z}_{q^n}$ in even dimensions, and zero in odd. The spectral sequence collapses, and so we see that

$$H^i(G; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}_{q^n} & \text{if } i \text{ is even, but } q \nmid i, \\ \mathbb{Z}_{pq^n} & \text{if } 2q \mid i, \\ 0 & \text{otherwise,} \end{cases}$$

and that restriction to the subgroup \mathbb{Z}_{q^n} is a cohomology isomorphism in dimensions below $2q$.

Lemma 5.1. *Let M be a closed, oriented four-manifold with $H_1(M)$ torsion-free, and $\chi(M) \neq 0$. If a nonabelian group of the form $G = C_p \rtimes C_q$ (with p and q prime) acts HTLL on M , then the \mathbb{Z} -coefficient BSS for the action collapses.*

Proof. Pick a representative subgroup $C_q \subset G$. If $q > 2$, then the target of every differential $d_r^{0,j}$ lies within the region where restriction to the cohomology of C_q is an isomorphism, and factoring through the spectral sequence for $H_{C_q}^*(M)$ shows that the given sequence collapses.

If $q = 2$, then the differential $d_4^{0,3} : H^0(G; H^3(M; \mathbb{Z})) \rightarrow H^4(G; H^0(M; \mathbb{Z})) \cong \mathbb{Z}_{2p}$ needs special consideration. But restriction to C_2 takes care of potential 2-torsion, and restriction to C_p rules out p -torsion. \square

Theorem 5.2. *Let M be a closed, oriented four-manifold with $H_1(M)$ nontrivial and torsion-free, $\chi(M) \neq 0$, and $b_2(M) \neq 0, 2$. The group $G = C_p \rtimes C_q$ (with p and q prime) admits no effective, HTLL action on M .*

Proof. Suppose such an action exists. By Lemma 5.1, $H_G^*(M; \mathbb{Z}) \cong H^*(G) \otimes H^*(M; \mathbb{Z})$. Let $u \in H^{2q}(G; \mathbb{Z})$ be a periodicity generator, and consider $S = \langle qu \rangle \subset H^*(G; \mathbb{Z})$. In this case, $\Sigma_S = \text{Fix}(C_p)$, and $S^{-1}H^d(G)$ is a \mathbb{Z}_{2q} -graded module which is isomorphic to \mathbb{Z}_p if $2q|d$, and trivial otherwise. It follows that

$$S^{-1}H_G^{\text{odd}}(\text{Fix}(C_p)) \cong \begin{cases} (\mathbb{Z}_p)^{b_1(M)} & \text{in degrees } d \equiv 1 \text{ and } d \equiv 3 \pmod{2q}, \\ 0 & \text{otherwise.} \end{cases}$$

However, $S^{-1}H_G^{\text{odd}}(\text{Fix}(C_p)) \cong S^{-1}H^{2q}(G; H^1(\text{Fix}(C_p)))$. Now, $H^1(\text{Fix}(C_p))$ is a \mathbb{Z} -free $\mathbb{Z}[C_q]$ -module, and hence equivalent to a sum of modules of cyclotomic, trivial, and permutation types. By Shapiro's lemma, $H^*(G; \mathbb{Z}[C_q]) \cong H^*(C_p; \mathbb{Z})$, and in the cases $d = 2qk$, the restriction $H^d(G; \mathbb{Z}) \rightarrow H^d(C_p; \mathbb{Z})$ is onto. It follows from the coefficient short exact sequence $0 \rightarrow \mathbb{Z}[\lambda] \rightarrow \mathbb{Z}[C_q] \rightarrow \mathbb{Z} \rightarrow 0$ that

$$H^i(G; \mathbb{Z}[\lambda]) = \begin{cases} \mathbb{Z}_p & \text{if } i \text{ is even and } 2q \nmid i, \\ \mathbb{Z}_q & \text{if } i = 2qk + 1, \\ 0 & \text{otherwise.} \end{cases}$$

so each cyclotomic summand in $H^1(\text{Fix}(C_p))$ contributes \mathbb{Z}_p to $S^{-1}H_G^*(\text{Fix}(C_p))$ in each odd degree d except $d \equiv 1 \pmod{2q}$. Each free summand makes a contribution in every odd degree, and each trivial summand makes a contribution when $d \equiv 1 \pmod{2q}$.

If $q > 2$, then $S^{-1}H_G^5(\text{Fix}(C_p)) = 0$, which rules out the presence of cyclotomic and permutation summands in $H^1(\text{Fix}(C_p))$. On the other hand, trivial summands cannot account for the nontriviality of $S^{-1}H_G^3(\text{Fix}(C_p))$. It follows that no actions can exist.

If $q = 2$, a similar analysis shows that the numbers of cyclotomic and trivial summands in $H_1(\text{Fix}(C_p))$ are equal, and it follows that the trace of the C_2 action on $H_1(\text{Fix}(C_p))$ is zero. Since $b_0(\text{Fix}(C_p)) + b_2(\text{Fix}(C_p)) = b_2(M) + 2$, and $b_2(M) \neq 0, 2$, $\text{Fix}(C_p)$ has either a single two-dimensional component, or its distinct components represent different elements of $H_2(M)$. So the two-dimensional components of $\text{Fix}(C_p)$ are individually G -invariant. It follows that the Lefschetz number of the C_2 -action on $\text{Fix}(C_p)$ is positive, and that a fixed point x for the entire group action exists on some two-dimensional component of $\text{Fix}(C_p)$. However, any 2-dimensional component F of $\text{Fix}(C_p)$ forms a proper subset of $\text{Fix}(C_p)$, and hence each such component represents a nontrivial class in $H_2(M; \mathbb{Z})$. Consideration of the local representation of the dihedral group G on $T_x(M)$ shows that the C_2 -action must reverse orientation on the 2-plane fixed by C_p , and hence send $[F]$ to $-[F]$, contradicting homological triviality.

□

A refinement of this argument rules out the larger metacyclic groups:

Theorem 5.3. *Let M be a closed, oriented four-manifold with $H_1(M)$ nontrivial and torsion-free, $\chi(M) \neq 0$, and $b_2(M) \neq 0, 2$. The group $G = C_p \rtimes C_{q^n}$ (with p and q prime, and $n > 1$) admits no effective, HTLL action on M .*

Proof. Let $G = \langle a, b \mid a^p = 1 = b^{q^n}, bab^{-1} = b^r \rangle$, where $r^q \equiv 1 \pmod{p}$, and begin by considering the index q cyclic subgroup $H = \langle ab^q \rangle$. The argument of the previous section shows that $\langle b^q \rangle$ acts on the spectral sequence for the C_p action, and hence we have a short exact sequence of $C_{q^{n-1}}$ -modules

$$0 \rightarrow H^4(C_p; H^1(M)) \rightarrow \mathbb{Z}_p \otimes H^1(\text{Fix}(a)) \rightarrow H^2(C_p; H^3(M)) \rightarrow 0,$$

where $C_{q^{n-1}}$ acts trivially on the outside terms. By Maschke's theorem, $\mathbb{Z}_p \otimes H^1(\text{Fix}(a))$ must have trivial $C_{q^{n-1}}$ -action. Hence H acts trivially on $H^1(\text{Fix}(C_p); \mathbb{Z}_p)$, and the C_{q^n} action on $\mathbb{Z}_p \otimes H^1(\text{Fix } C_p)$ factors through a $G/H \cong C_q$ -action.

Now consider the spectral sequence of the G -action on M . It is not evident *a priori* that it collapses, but factoring through BSS for the C_p -action shows that all p -torsion survives, and it follows again that with $S = \langle qu \rangle$, we still have

$$S^{-1}H_G^{\text{odd}}(\text{Fix}(C_p)) \cong \begin{cases} (\mathbb{Z}_p)^{b_1(M)} & \text{in degrees } d \equiv 1 \text{ and } d \equiv 3 \pmod{2q}, \\ 0 & \text{otherwise.} \end{cases}$$

The rest of the argument proceeds exactly as before. □

Finally, we turn our attention to the quaternion group

$$Q_8 = \langle h, x \mid h^4 = x^4 = 1, xhx^{-1} = h^{-1}, h^2 = x^2 \rangle.$$

Q_8 has three cyclic subgroups of order 4: $\langle h \rangle$, $\langle x \rangle$, and $\langle xh \rangle$, all of which intersect in the central $\langle h^2 \rangle \cong C_2$.

Theorem 5.4. *Let M be a closed, oriented four-manifold with $H_1(M)$ nontrivial and torsion-free, and $\chi(M) \neq 0$. The group Q_8 admits no effective, HTLL action on M .*

Proof. Suppose Q_8 were to act in the stated manner. Denote the three copies of $C_4 \subset Q_8$ by K_1, K_2, K_3 , and consider the action of a subgroup K_i . The spectral sequence for the K_i -action (with \mathbb{Z}_2 coefficients) may not collapse, but by Proposition 3.1, the only potentially nonzero differential is d_2 from row 2 to row 1. In particular, row 3 consists of permanent cocycles.

Now, examination of the L-H-S spectral sequence (for example) shows that the restriction map $H^i(C_4; \mathbb{Z}_2) \rightarrow H^i(C_2; \mathbb{Z}_2)$ is trivial for every $i > 0$. Hence if u generates $H^2(C_4; \mathbb{Z}_2)$, and $S = \langle u \rangle$, then Σ_S consists only of points fixed by the entire group C_4 . Hence, by the Localization Theorem, $b_1(\text{Fix}(K_i)) \geq b_1(M)$.

But each $K_i \subset Q_8$ has the center C_2 as a subgroup, and clearly, $\text{Fix}(K_i) \subset \text{Fix}(C_2)$. Let F be the set of two-dimensional components of $\text{Fix}(C_2)$, and for each i , let F_i be the corresponding set of two-dimensional components of $\text{Fix}(K_i)$. Let $V = H_1(F; \mathbb{Z}_2)$, and $V_i = H_1(F_i; \mathbb{Z}_2) \subset V$. By a dimension count, it follows that some pair V_i, V_j must have nontrivial intersection, and hence that F_i and F_j have at least one two-dimensional component in common. But such a component is necessarily fixed by all of Q_8 . This is impossible, as Q_8 cannot act faithfully on a two-dimensional slice disk. \square

We have now eliminated groups of rank ≥ 2 and all nonabelian groups, so the main theorem follows.

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